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The Asymptotic Distribution of Zeros of Minimal Blaschke Products

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1. INTRODUCTION

Let D be the open unit disc in the complex plane. A Blaschke product B of degree n, $n \geqslant 1$, is an analytic function of the form

$$B(z) = \lambda \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}; \qquad |\lambda| = 1, \quad a_k \in D, \quad k = 1, ..., n. \tag{1}$$

We let B_n denote the set of all Blaschke products of degree n or less; B_n is compact in the topology of uniform convergence on compact subsets of D. Let W be a non-negative continuous function on D and let E be a compact subset of D. In this paper we shall be concerned with the asymptotic distribution, as $n \to \infty$, of the zeros of the solutions to the extremal problem

$$\min_{B \in B_n} \|B W^n\|_E, \tag{2}$$

where the norm $\|\cdot\|_E$ is the sup norm over E. Such minimization problems arise, for instance, in the theory of n-widths of sets of analytic functions; see [3] and [18] where $W \equiv 1$. Our analysis goes beyond Blaschke products and the unit disc and leads us to results about the asymptotic behavior of the zeros of other functions on D and on other domains. Fekete and Walsh [4] and Blatt, Saff, and Simkani [1] explore some of the same problems raised here but in the more classical setting of minimal polynomials with

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weight $W \equiv 1$. There the result is that, under rather general conditions, the nth root of the minimum converges to the transfinite diameter of E and by placing the mass $1/n$ at each of the zeros of the nth minimal polynomial, one obtains a sequence of measures that behaves asymptotically like the equilibrium measure for the set E, provided E has connected complement and empty interior. We will see counterparts to these results in this paper. Mhaskar and Saff [13] first explored weighted approximation problems of the type considered here for subsets E of the real line; in [14] they investigate similar asymptotic behavior for extremal polynomials in the complex plane. The asymptotic distributions of certain extremal point sequences subject to an external field were also studied in the works of Górski [5] and Kleiner [8].

In Section 2 we consider a minimal energy problem in the presence of an external field, where the energy is with respect to the Green potential of a bounded domain Ω. Section 3 contains the main result on the asymptotic distribution of zeros of optimizing sequences. Some examples illustrating the main result of Section 3 are included.

2. EQUILIBRIUM MEASURES

Portions of this section are drawn from the comprehensive book [15] by Saff and Totik.

Throughout this paper, Ω is a bounded domain in the complex plane that is regular for the Dirichlet problem and $g(z; \zeta)$ is the Green function for Ω with singularity at ζ. We shall use the standard terminology that a property holds *quasi-everywhere*, and write q.e., if the set on which the property does not hold has logarithmic capacity zero.

The following well-known facts are needed at several points later; (b) is a version of the "lower envelope theorem;" and (c) is called the "principle of descent;" see [6; p. 62] and [15; Theorems 0.1.4, I.6.9, and I.6.8, respectively].

PROPOSITION 1. *Let $\{\mu_n\}$ be a sequence of positive measures supported on some compact subset F of Ω and suppose that $\{\mu_n\}$ converges weak-star to μ.*

(a) *If H is lower semi-continuous on F, then* $\int H\, d\mu \leqslant \liminf \int H\, d\mu_n$.

(b) $\int g(z; \zeta)\, d\mu(\zeta) = \liminf \int g(z; \zeta)\, d\mu_n(\zeta)$ q.e. *on* Ω.

(c) *If $z_n \to z$, then* $\int g(z; \zeta)\, d\mu(\zeta) \leqslant \liminf \int g(z_n; \zeta)\, d\mu_n(\zeta)$.

Let S be a compact subset of Ω. The *holomorphic hull* of S relative to Ω, written $\hat{S}$, is the union of S and all those components of the complement of S relative to the Riemann sphere that do not contain points of the boundary of Ω. Equivalently,

$$\hat{S} = \{z \in \Omega : |f(z)| \leqslant \|f\|_s \text{ for all functions } f \text{ that are analytic on } \Omega\}.$$

The set of probability measures on a compact subset E is denoted by $\mathscr{P}(E)$. $S(v)$ denotes the (closed) support of a measure v and ∂A denotes the topological boundary of a set A. Finally, Q is an extended real-valued function on E that is lower semi-continuous on E, $Q > -\infty$ on E, and $Q < \infty$ q.e. on E. We assume throughout that $E \subset \Omega$.

Define

$$V_Q = \inf_{\mu \in \mathscr{P}(E)} \int_E \int_E [g(z; \zeta) + Q(z) + Q(\zeta)] \, d\mu(z) \, d\mu(\zeta) \tag{3}$$

and

$$T_Q = \inf_{\alpha \in \mathscr{P}(E)} \sup_{z \in E} \int_E [g(z; \zeta) + Q(\zeta)] \, d\alpha(\zeta) \tag{4}$$

If $Q \equiv 0$ in (3) and (4), then we denote the corresponding quantities by V_E and T_E, respectively. The following result is standard; see [7] or [15], for instance.

PROPOSITION 2. *Suppose E is a compact set of positive capacity. Then there is a unique measure $\mu_E \in \mathscr{P}(E)$ such that*

(a) $\int_E \int_E g(z; \zeta) \, d\mu_E(z) \, d\mu_E(\zeta) = V_E$ *and* $T_E = V_E$;

(b) $\int_E g(z; \zeta) \, d\mu_E(\zeta) \leqslant T_E$ *for all* $z \in \Omega$;

(c) *the support* $S(\mu_E)$ *of* μ_E *lies in* $\partial \hat{E}$ *and* $\partial \hat{E} \setminus S(\mu_E)$ *has capacity zero*;

(d) $\int_E g(z; \zeta) \, d\mu_E(\zeta) = T_E$ *q.e. on* $\partial \hat{E}$ *and for all* $z \in \operatorname{int} \hat{E}$.

DEFINITION. The measure μ_E from Proposition 2 is called the *Green equilibrium measure for E*.

The analogue of Proposition 2 in the presence of an external field Q is the following.

PROPOSITION 3. *Suppose the compact set E has positive logarithmic capacity. Then there is a unique measure $\mu_Q \in \mathscr{P}(E)$ such that*

(a) $V_Q = \int_E \int_E [g(z; \zeta) + Q(z) + Q(\zeta)] \, d\mu_Q(z) \, d\mu_Q(\zeta)$;

(b) $\int_E [g(z; \zeta) + Q(z) + Q(\zeta)] \, d\mu_Q(\zeta) \geqslant V_Q$ *q.e. on* E; *equality holds q.e. on the support* $S(\mu_Q)$ *of* μ_Q;

(c) *the quantities* V_Q *and* T_Q *are related by* $V_Q = T_Q + \int_E Q\,d\mu_Q$;

(d) *if* μ_S *is the Green equilibrium measure for the support* $S = S(\mu_Q)$ *of* μ_Q, *then* $\int [g(z;\zeta) + Q(\zeta)]\,d\mu_S(\zeta) \leqslant T_Q$ *for all* $z \in \Omega$.

Proof. It is easy to see that V_Q defined in (3) is finite; indeed, $V_Q > -\infty$ follows from the lower boundedness of Q on E and $V_Q < \infty$ follows by taking $\mu = \mu_{E_n}$, where n is so large that $E_n := \{z \in E \colon Q(z) \leqslant n\}$ has positive capacity. Proofs of (a)–(c) may be found in [15; Chapter II, Section 5]. To prove (d) we integrate the equality in (b) with respect to μ_S and then apply (c); this gives

$$\iint g(z;\zeta)\,d\mu_Q(\zeta)\,d\mu_S(z) + \int Q(z)\,d\mu_S(z) = V_Q - \int Q\,d\mu_Q = T_Q.$$

However, by Proposition 2, $\int g(z;\zeta)\,d\mu_S(z) = T_S$ q.e. on S where T_S is short for $T_{S(\mu_Q)}$. Hence,

$$T_S + \int Q\,d\mu_S = T_Q. \tag{5}$$

Set $u(z) = \int [g(z;\zeta) + Q(\zeta)]\,d\mu_S(\zeta)$. Then by Proposition 2(b) and (5)

$$u(z) \leqslant T_S + \int Q\,d\mu_S = T_Q, \qquad \text{for all} \quad z \in \Omega.$$

This establishes (e).

Definition. The measure μ_Q given in Proposition 3 is called the *Green equilibrium measure for* Q *on* E.

We now introduce the analogue of the transfinite diameter in the presence of an external field. This exposition is based on material from [15, Chapter III, Section 1].

Definition. Let E be a compact set in Ω and let Q satisfy the hypotheses at the beginning of Section 2. Set

$$\delta_n^Q = \max_{z_1, \ldots, z_n \in E} \left(\prod_{1 \leqslant i < j \leqslant n} \exp[-g(z_i, z_j) - Q(z_i) - Q(z_j)] \right)^{2/n(n-1)}. \tag{6}$$

We remark that in the special case when Ω is the open unit disk and there is no external field ($Q \equiv 0$), then the extremal points for (6) are nothing but the Tsuji points which play an important role in approximation theory (see [9], [12]).

The basic facts about the sequence $\{\delta_n^Q\}$ are contained in the next proposition.

PROPOSITION 4. *Let E be a compact subset of Ω and let δ_n^Q be given by* (6). *Then the following assertions hold:*

(a) $\delta_n^Q \geqslant \delta_{n+1}^Q$, $n = 1, 2, 3, ...$;

(b) $\lim_{n\to\infty} \log \delta_n^Q = -V_Q$;

(c) *if $\{\zeta_1^*, ..., \zeta_n^*\}$ is any optimal set for δ_n^Q and γ_n is the measure obtained by placing the weight $1/n$ at each of the points $\{\zeta_1^*, ..., \zeta_n^*\}$, then the sequence of measures $\{\gamma_n\}$ converges weak-star to the Green equilibrium measure μ_Q for Q on E*;

(d) $\lim_{n\to\infty} \int Q\, d\gamma_n = \int Q\, d\mu_Q$.

We conclude this section by stating a result concerning the well-developed concept of "balayage" with respect to the Green function (cf. [6, 10]). Because the proof follows from well-known techniques in potential theory, we omit the details (see, e.g., [15, Chapter II, Section 4]).

PROPOSITION 5. *Let E be a compact set with positive logarithmic capacity and suppose that $\beta \in \mathscr{P}(E)$ is either* (*a*) *supported on the interior of E or* (*b*) *has finite energy. Then there is a unique positive measure $[\beta]_b$ on ∂E of total mass one and finite energy such that for all $z \in \Omega \backslash E$ and quasi-every $z \in \partial E$*

$$\int_{\partial E} g(z; \zeta)\, d[\beta]_b(\zeta) = \int_E g(z; \zeta)\, d\beta(\zeta). \tag{7}$$

DEFINITION. The measure described in Proposition 5 is called the *sweep* (*or balayage*) *of β to ∂E relative to the Green function.*

3. ASYMPTOTIC DISTRIBUTION OF OPTIMIZING POINTS

We return to the minimization problem (2) posed in the introduction. A Blaschke product of degree n on the unit disc D with zeros at $\zeta_1, ..., \zeta_n$ has modulus $|B(z)| = \exp(-\sum_{k=1}^n g(z; \zeta_k))$ where $g(z; \zeta)$ is the Green function for D with pole at ζ. If W is continuous and $|W| = \exp(-Q)$, then

$$\min_{B \in B_n} \{\log \|BW^n\|_E\} = \min \left\{ \max_{z \in E} \left[-\sum_{k=1}^n g(z; \zeta_k) - nQ(z) \right] : \zeta_1, ..., \zeta_n \in D \right\}$$
$$= -\max \left\{ \min_{z \in E} \left[\sum_{k=1}^n g(z; \zeta_k) + nQ(z) \right] : \zeta_1, ..., \zeta_n \in D \right\}.$$

Thus, to solve (2) we must make the minimum over E of the sum of n Green functions plus $nQ(z)$ as large as possible and then determine the asympotic behavior of this quantity as $n \to \infty$. Here are three examples illustrating some of the possible outcomes of this sort of minimization.

EXAMPLE 1. Let E be any compact subset of the open unit disc D. It is known that the zeros of the minimizing Blaschke product lie in the convex hull of E with respect to the hyperbolic geometry in D; see [17]. If E has no interior and is convex with respect to the hyperbolic geometry, then the zeros lie in E. For instance, if E is a subset of $(-1, 1)$, then the zeros of the minimizing Blaschke product will lie in the smallest closed interval $[a, b]$ that contains E. In particular, if E is itself a closed interval in $(-1, 1)$, then the zeros lie in E.

EXAMPLE 2. On D, take $E = \{z : |z| \leqslant r\}$ where $0 < r < 1$ and $Q \equiv 0$. The Blaschke product of degree n of minimal sup norm over E is z^n; see Theorem 9 of [3]. Therefore all n zeros of the minimizing Blaschke product lie at the origin.

EXAMPLE 3. Again on D, we fix a Blaschke product B_0 of degree $N \geqslant 2$ and $r \in (0, 1)$. We take $E = \{z \in D : |B_0(z)| \leqslant r\}$ and $Q = -\log |B_0|$. When r is small, E has as many components as B_0 has distinct zeros; as r increases these components coalesce until there is just one when r is near 1. If $n = kN$, then the Blaschke product B of degree n that minimizes $\|BB_0^n\|_E$ is just B_0^k. Here is an outline of the proof, which is much like Theorem 9 of [3]. Since B_0^k has degree $kN = n$, it is a competitor for the minimal Blaschke product of degree n and $|B_0^k B_0^n| \equiv r^{k+n}$ on ∂E. Suppose $B \neq B_0^k$ has degree n or less and $|BB_0^n| \leqslant r^{k+n}$ on ∂E. We may multiply each of B and B_0 by a unimodular scalar without altering the minimality of B, so there is no loss in assuming that $B_0(1) = B(1)$. We have $|B| \leqslant r^k$ on ∂E. Let $\varepsilon > 0$; apply Rouche's theorem to $(1+\varepsilon) B_0^k - B$ and $(1+\varepsilon) B_0^k$ first on ∂E and then on the unit circle $\mathbf{T}$. We conclude that $(1+\varepsilon) B_0^k = B$ at n points of the interior of E and at exactly n points in D, which obviously are then the same as those already found in the interior of E. Letting $\varepsilon \to 0$, we see that $B = B_0^k$ at n points of E and at no other points of D. On the other hand, $B - B_0^k$ vanishes at the point 1 and so when ε is sufficiently small, $(1+\varepsilon) B_0^k - B$ must have a zero near 1 that evidently must lie outside the closed unit disc, say at w, $|w| > 1$. By reflection, $(1+\varepsilon) B(w^*) = B_0^k(w^*)$ where w^* is the reflection of w over $\mathbf{T}$. This implies that $(1+\varepsilon) B - B_0^k$ has at least $n+1$ zeros in D, a contradiction. Hence, the zeros of the minimal norm Blaschke product of degree $n = kN$, are the N zeros of B_0, each repeated k times.

To attack the minimization problem in (2) and its extensions to other domains, we make the following definition.

DEFINITION.

$$b_n = b_n(E; Q) = \sup \left\{ \min_{z \in E} \left[\frac{1}{n} \sum_{k=1}^{n} g(z; \zeta_k) + Q(z) \right] : \zeta_1, ..., \zeta_n \in \Omega \right\}. \tag{8}$$

The first result gives the asymptotic behavior of b_n.

PROPOSITION 6. *If E has positive capacity, then* $\lim_{n \to \infty} b_n = T_Q$.

Proof. Suppose there is a unit measure β with compact support in Ω and a number $T' > T_Q$ with

$$\int g(z; \zeta) \, d\beta(\zeta) + Q(z) \geqslant T', \qquad \text{for all} \quad z \in E. \tag{9}$$

Integrate both sides of (9) with respect to μ_S, the Green equilibrium measure for $S = S(\mu_Q)$. This gives

$$T' \leqslant \iint [g(z; \zeta) + Q(z)] \, d\mu_S(z) \, d\beta(\zeta) \leqslant T_Q,$$

which is a contradiction. Hence,

$$\max_{\beta \in \mathscr{P}(\Omega)} \min_{z \in E} \left(\int g(z; \zeta) \, d\beta(\zeta) + Q(z) \right) \leqslant T_Q$$

so that $b_n \leqslant T_Q$ for all n. To see that $b_n \to T_Q$, we use Proposition 4. Let $\{z_1, ..., z_{n+1}\}$ be an optimal set for δ_{n+1}^Q and let γ_{nj} be the measure obtained by placing the mass $1/n$ at the points z_i, $i \neq j$. From (c) of Proposition 4, the weak-star limit of $\{\gamma_{nj}\}$ is the equilibrium measure μ_Q for Q on E. We then have

$$\begin{aligned}
&\sum_{i=1;\, i \neq j}^{n+1} [g(z_j; z_i) + Q(z_i) + Q(z_j)] \\
&\quad = \min_{z \in E} \sum_{i=1;\, i \neq j}^{n+1} [g(z; z_i) + Q(z)] + \sum_{i=1;\, i \neq j}^{n+1} Q(z_i) \\
&\quad \leqslant n b_n + n \int Q \, d\gamma_{nj}.
\end{aligned}$$

Now sum on j from 1 to $n+1$ and divide both sides by $n(n+1)$ to get

$$-\log \delta_{n+1}^{Q} \leqslant b_n + \frac{1}{n+1} \sum_{j=1}^{n+1} \int Q \, d\gamma_{nj}.$$

It follows from (b) and (d) of Proposition 4 that $V_Q \leqslant \liminf b_n + \int Q \, d\mu_Q$. Equivalently, $\liminf b_n \geqslant T_Q$.

Definition. A sequence of n-tuples $\{\zeta_{1n}, ..., \zeta_{nn}\}$, $n = 1, 2, 3, ...,$ is *asymptotically optimal* if

$$\lim_{n \to \infty} \min_{z \in E} \left(\frac{1}{n} \sum_{k=1}^{n} g(z; \zeta_{kn}) + Q(z) \right) = T_Q. \tag{10}$$

The appearance of the quantity T_Q as the limit of the numbers b_n raises the possibility that there is a connection between the Green equilibrium measure for Q on E and the limit of the measure obtained by placing the mass $1/n$ at each of the n points of an asymptotically optimal sequence. This has already proved to be the case when the asymptotically optimal sequence is obtained from the sequence described in Proposition 4. In fact, let us review Example 3 in this light.

Example 4. Let B_0 and E be as in Example 3. Let β be the measure formed by placing the mass $1/N$ at each of the N zeros of B_0 and let $[\beta]_b$ be its sweep to ∂E. Then the sum of $Q = -\log |B_0|$ and the Green potential of $[\beta]_b$ is equal to $-2 \log r$ quasi-everywhere on ∂E and exceeds this quantity on the interior of E (by the minimum principle for superharmonic functions). By Theorem II.5.12 of [15], $[\beta]_b$ is the Green equilibrium measure for Q on E.

In fact, the primary characteristics of Example 3 may be extended quite generally. Let Ω be any domain that is regular for the Dirichlet problem and let $\zeta_1, ..., \zeta_N$ be points (not necessarily distinct) of Ω. Fix a positive number M and define

$$E = \left\{ z \in \Omega : \sum_{j=1}^{N} g(z; \zeta_j) \geqslant M \right\}.$$

We take $Q(z) = \sum_{j=1}^{N} g(z; \zeta_j)$. Let β be the measure formed by placing the mass $1/N$ at each of the N points $\zeta_1, ..., \zeta_N$ and let $[\beta]_b$ be its sweep to ∂E. Then the sum of Q and the Green potential of $[\beta]_b$ is equal to $(1 + 1/N)M$ quasi-everywhere on ∂E and exceeds this quantity on the interior of E. Again, Theorem II.5.12 of [15], shows that $[\beta]_b$ is the Green equilibrium measure for Q on E.

Example 4 shows that sweeping can establish a connection among the measures associated to an asymptotically optimal sequence and the Green equilibrium measure for Q on E, which is supported on ∂E. This leads us to the following construction.

Let μ_Q be the Green equilibrium measure for Q on E and let $S = S(\mu_Q)$ be its support. For an asymptotically optimal sequence, we define a measure ρ_n as

$$\rho_n = \frac{1}{n} \sum_{j=1}^{n} \sigma_{jn}, \tag{11}$$

where σ_{jn} is the measure given by

$$\sigma_{jn} = \begin{cases} \text{the point mass at } \zeta_{jn} \text{ if } \zeta_{jn} \notin \operatorname{int} \hat{S}, \\ \text{the sweep to } \partial\hat{S} \text{ of the unit point mass at } \zeta_{jn} \text{ if } \zeta_{jn} \in \operatorname{int} \hat{S}. \end{cases}$$

Above and in what follows, "sweep" is understood to mean the sweep relative to the Green function of Ω, as described in Proposition 5 at the end of Section 2.

The main result of this section is the following.

THEOREM 7. *The sequence* $\{\rho_n\}$ *of measures defined in* (11) *converges weak-star to* $[\mu_Q]_b$, *the sweep of* μ_Q *to* $\partial\hat{S}$, *where* $S = S(\mu_Q)$.

Proof. Let μ_S be the (unweighted) Green equilibrium measure for $S = S(\mu_Q)$ and set

$$v(z) = \int (g(z; \zeta) + Q(\zeta))\, d\mu_S(\zeta).$$

We know from Proposition 3(d) that $v(z) \leqslant T_Q$ for all $z \in \Omega$. Further, $v(z)$ is harmonic on $\Omega \backslash \partial\hat{S}$ and is identically equal to (the constant) $\int Q\, d\mu_S$ on the boundary of Ω. Thus, $v(z) < T_Q$ for all z in the complement of $\hat{S}$; moreover, v is identically constant on the interior of $\hat{S}$. Fix $\delta > 0$ and let

$$E_\delta = \{z \in \Omega \colon v(z) \geqslant T_Q - \delta\}.$$

Let $A = A(\delta)$ be those indices j, $1 \leqslant j \leqslant n$, for which $\zeta_{jn} \notin E_\delta$ and let $D = D(\delta)$ be the remaining indices. We denote the cardinalities of A, D by $A^{\#}$, $D^{\#}$, respectively. Since $\{\zeta_{jn}\}$ is asymptotically optimal, there are numbers $\varepsilon_n \to 0$, such that

$$T_Q - \varepsilon_n \leqslant \frac{1}{n} \sum_{k=1}^{n} g(z; \zeta_{kn}) + Q(z), \qquad z \in E.$$

Multiply both sides of this inequality by n and then integrate with respect to μ_S. This gives

$$\begin{aligned} n(T_Q - \varepsilon_n) &\leqslant \sum_{j=1}^{n} \int [g(z; \zeta_{jn}) + Q(z)]\, d\mu_S(z) \\ &= \sum_{j=1}^{n} v(\zeta_{jn}) = \sum_{j \in A} + \sum_{j \in D} \\ &\leqslant (A^{\#})(T_Q - \delta) + (D^{\#})\, T_Q \\ &= nT_Q - \delta A^{\#}. \end{aligned}$$

Hence, $\varepsilon_n \geqslant \delta A^{\#}/n$. We conclude that

$$\lim_{n \to \infty} \frac{A^{\#}}{n} = 0. \tag{12}$$

Let ρ be a weak-star cluster point of the sequence $\{\rho_n\}$ defined in (11). Then by (12) the portion of ρ_n that is contributed by points ζ_{jn} that lie at a positive distance from $\hat{S}$ is vanishingly small as $n \to \infty$. Hence, ρ is supported on $\partial \hat{S}$.

We have

$$\frac{1}{n} \sum_{k=1}^{n} g(z; \zeta_{kn}) = \int g(z; \zeta)\, d\rho_n(\zeta) \qquad \text{q.e. on } \partial \hat{S}$$

by the definition of ρ_n and the properties of sweeping. By the lower envelope theorem

$$\liminf \int g(z; \zeta)\, d\rho_n(\zeta) = \int g(z; \zeta)\, d\rho(\zeta) \qquad \text{q.e. on } \Omega.$$

Hence, $T_Q - Q(z) \leqslant \int g(z; \zeta)\, d\rho(\zeta)$ q.e. on $\partial \hat{S}$. Moreover, by Proposition 3(d),

$$\begin{aligned} T_Q &\leqslant \int \left[\int g(z; \zeta)\, d\rho(\zeta) + Q(z) \right] d\mu_S(z) \\ &= \iint [g(z; \zeta) + Q(z)]\, d\mu_S(z)\, d\rho(\zeta) \leqslant T_Q. \end{aligned}$$

It follows that

$$T_Q = \int g(z; \zeta)\, d\rho(\zeta) + Q(z) \qquad \text{a.e.} \quad \mu_S. \tag{13}$$

However, we know that the support of μ_S is all of $\partial\hat{S}$ except possibly for a set of capacity zero. The right-hand side of (13) is lower semi-continuous and because μ_S is an equilibrium measure, every set of positive μ_S-measure has positive capacity. Hence

$$T_Q \geqslant \int g(z; \zeta)\, d\rho(\zeta) + Q(z) \qquad \text{everywhere on } \partial\hat{S}. \tag{14}$$

Now integrate (14) with respect to ρ; we obtain

$$T_Q \geqslant \iint g(z; \zeta)\, d\rho(\zeta)\, d\rho(z) + \int Q\, d\rho.$$

Hence, ρ has finite Green energy. We also learn that for quasi-every $z \in \partial\hat{S}$

$$\int_{\partial\hat{S}} g(z; \zeta)\, d\rho(\zeta) = \int_S g(z; \zeta)\, d\mu_Q(\zeta) = \int_{\partial\hat{S}} g(z; \zeta)\, d[\mu_Q]_b\,(\zeta). \tag{15}$$

Because ρ has finite energy, we may apply the Principle of Domination to conclude that

$$\int_{\partial\hat{S}} g(z; \zeta)\, d\rho(\zeta) \leqslant \int_{\partial\hat{S}} g(z; \zeta)\, d[\mu_Q]_b\,(\zeta), \qquad z \in \Omega.$$

However, the measure $[\mu_Q]_b$ also has finite energy and so the opposite inequality holds, as well. Thus, the Green potentials of the measures ρ and $[\mu_Q]_b$ agree in Ω and therefore the measures coincide; see [15; Theorem II.5.3].

COROLLARY 8. *When $Q \equiv 0$ the measures ρ_n converge weak-star to the Green equilibrium measure of E. In particular, if $E = \partial\hat{E}$, then placing the mass $1/n$ at each of the points ζ_{jn}, $j = 1, ..., n$, produces asymptotically the Green equilibrium measure of E.*

The following two examples illustrate the conclusion of Theorem 7.

EXAMPLE 5. Let $\Omega = D$, the open unit disc, let E be the closed interval $[-a, a]$, $0 < a < 1$ and take $Q \equiv 0$. It is known that

$$d\mu_E(x) = \frac{1}{2K} \frac{dx}{\sqrt{(a^2 - x^2)(1 - a^2x^2)}}, \qquad -a < x < a$$

where

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-a^4t^2)}};$$

see [15; Example II.5.14]. The n zeros of the Blaschke product of degree n of smallest sup norm on the interval $[-a, a]$ lie in this interval; see Example 1. By Theorem 7 they are distributed asymptotically to give the measure μ_E.

EXAMPLE 6. Let Ω be any bounded domain that is regular for the Dirichlet problem, let ν be a positive measure of total mass one with compact support E within Ω and finite Green energy, and let $U = U_G^\nu$ be its Green potential; that is,

$$U(z) = \int g(z; \zeta)\, d\nu(\zeta), \qquad z \in \Omega.$$

We take $Q = -cU$, where $c \in (0, 1)$. It is known, see [15; Example III.5.15], that the Green equilibrium measure for Q on E is

$$\mu_Q = c\nu + (1-c)\,\mu_E, \tag{16}$$

where μ_E is the (unweighted) Green equilibrium measure for E. One illustration of this is obtained by taking $\Omega = D$ and E as in Example 5. Another example may be obtained from the construction in Example 4. Specifically, let $\zeta_1, ..., \zeta_N$ be N (distinct) points of Ω and let

$$E = \left\{ z \in \Omega : \sum_{k=1}^{N} g(z; \zeta_k) \geqslant M \right\}.$$

We take ν to be, say, Lebesgue area measure on E. As in Example 4, the (unweighted) Green equilibrium measure μ_E for E is $1/N$ times the sum of the Green sweep to ∂E of the point mass measure at ζ_k, $k = 1, ..., N$. Making use of (16), we obtain μ_Q.

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REFERENCES

1. H.-P. Blatt, E. B. Saff, and M. Simkani, Jentzsch–Szegő type theorems for the zeros of best approximants, *J. London Math. Soc.* **38** (1988), 307–316.

2. S. D. Fisher, "Function Theory on Planar Domains," Wiley, New York, 1983.
3. S. D. Fisher and C. A. Micchelli, The n-width of sets of analytic functions, *Duke Math. J.* **47** (1980), 789–801.
4. M. Fekete and J. L. Walsh, On the asymptotic behavior of polynomials with extremal properties, and of their zeros, *J. Anal. Math.* **IV** (1954), 49–87.
5. J. Górski, Une remarque sur la méthode des points extrémaux de F. Leja, *Ann. Polo. Math.* (1959), 63–69.
6. L. L. Helms, "Introduction to Potential Theory," Wiley, New York, 1963.
7. E. Hille, "Analytic Function Theory," Vol. II, Ginn and Company, Boston, 1962.
8. W. Kleiner, Degree of convergence of the extremal points method for Dirichlet's problem in the space, *Colloquium Math.* **12** (1964), 41–52.
9. H. Kloke, "Punktsysteme mit extremalen Eigenschaften für ebene Kondensatoren," Doctoral thesis, Univ. Dortmund, 1984.
10. N. S. Landkof, "Foundations of Modern Potential Theory," Die Grundlehren der Mathematische Wissenschaften, Vol. 180, Springer-Verlag, Berlin, 1972.
11. A. L. Levin and E. B. Saff, Szegő type asymptotics for minimal Blaschke products, *in* "Progress in Approximimation Theory" (A. A. Gonchar and E. B. Saff, Eds.), pp. 105–126, Springer-Verlag, Berlin/New York, 1992.
12. K. Menke, On Tsuji points in continuum, *Complex Variables Theory Appl.* **2** (1983), 165–175.
13. H. N. Mhaskar and E. B. Saff, Where does the sup norm of a weighted polynomial live?, *Const. Approx.* **1** (1985), 71–91.
14. H. N. Mhaskar and E. B. Saff, The distribution of zeros of asymptotically extremal polynomials, *J. Approx. Theory* **65** (1991), 279–300.
15. E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Springer-Verlag, Heidelberg, 1997.
16. M. Tsuji, "Potential Theory in Modern Function Theory," Maruzen, Tokyo, 1959.
17. J. L. Walsh, Note on the location of zeros of extremal polynomials in the non-Euclidean plane, *Serbian Acad. Sci.* **IV** (1952), 157–160.
18. H. Widom, Rational approximation and n-dimensional diameter, *J. Approx. Theory* **5** (1972), 343–361.